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Dialetheism and Logicism

In this paper, I will consider the possible relationships between dialetheism and the logicist project. First, I will sketch how dialetheism might be a natural response to Russell's paradox, one of the key obstacles for logicism. Next, I will elaborate on the details of dialetheism and argue that it provides a viable theoretical perspective. Then, I will consider the objection to dialetheism posed by the principle of explosion – in other words, "contradictions entail everything." I will respond to the objection by outlining a paraconsistent logic in which logicism is not necessarily doomed. Finally, I will consider the merits and disadvantages of dialetheism and argue that it is a valuable and viable perspective.

Dialetheism is essentially the view that there are true contradictions (called *dialetheias*); in other words, dialetheism denies that a statement cannot be both true and false. This intuitive principle – symbolically, $\neg(A \wedge \neg A)$ – is known as the Law of Non-Contradiction (LNC).[1] The first good articulation of the principle can be traced back to Aristotle (*Metaphysics* Γ), although the consensus is that his "arguments are surprisingly poor."[2] Since then, the LNC has often been taken as "*the* basic, indemonstrable 'first principle.'"[3] Avicenna's commentary on the *Metaphysics* illustrates the common view that the LNC "and their like are among the things that do not require our elaboration." Avicenna's words for "the obdurate" are quite facetious: "he must be subjected to the conflagration of fire, since 'fire' and 'not fire' are one. Pain must be inflicted on him through

[1] Note that this is different than the Law of Excluded Middle, $A \vee \neg A$.
[2] Priest and Berto, "Dialetheism."
[3] Horn, "Contradiction."

beating, since 'pain' and 'no pain' are one. And he must be denied food and drink, since eating and drinking and the abstention from both are one [and the same]."[4]

Yet a number of philosophers over the past century have indeed denied the Law of Non-Contradiction, foremost among them Graham Priest. Why have they sought to do so? One answer lies in the troubles faced by the logicist project. Logicism seeks to reduce all of mathematics to logic and definitions. It was pioneered by Gottlob Frege and developed by, among others, Richard Dedekind and Bertrand Russell. Briefly, Frege's aim was to give definitions from which all of arithmetic could logically proceed. His first attempt was Hume's Principle (HP): the number belonging to the concept C is identical to the number belonging to the concept D if and only if C is equinumerous with D. This view is attractive for many reasons. For example, HP is simple and can easily be used to define the natural numbers. Unfortunately, HP has serious problems. For instance, how can one use HP to deal with nonsensical sentences such as "the number belonging to the concept C is identical to Julius Caesar"? In response to such issues, Frege proposed a new definition known as Basic Law V: the extension of the concept C and the extension of the concept D are the same if and only if C and D have the same things falling under them. Since Julius Caesar is not an extension, Basic Law V easily shows that such a nonsensical sentence is false.

Yet this new definition faces an even bigger issue: Russell's paradox. In modern set-theoretic language this paradox is stated as follows. Let the Russell set *Rs* be the set of all sets that don't contain themselves. For instance, the set of all people is not an element of *Rs*; the set of all sets is. What about *Rs* itself? If *Rs is* a member of *Rs*, *Rs doesn't* contain itself and hence *Rs isn't* a member of *Rs*. On the other hand, if *Rs isn't* a member of *Rs*, *Rs does* contain itself and hence *Rs*

[4] Avicenna, *The Metaphysics of* The Healing, I.8 53.13–15 (p. 43).

is a member of Rs. Symbolically, if $Rs = \{x \mid x \notin x\}$ then $Rs \in Rs \Leftrightarrow Rs \notin Rs$. In short, Russell's paradox illustrates a serious inconsistency that lies within Basic Law V. Specifically, the formulation of Basic Law V entails a comprehension schema: roughly, there exists a set that contains all and only elements with a certain predicate, where there are infinitely many axioms for as many predicates.[5] What happens if that predicate is $x \notin x$? We get the Russell set and hence contradiction. In other words, the Russell set demonstrates that naive comprehension leads to contradiction. Basic Law V, as formulated by Frege, is inconsistent.

Frege tried to repair Basic Law V, but after a few attempts he abandoned logicism as a project doomed to failure. Russell and numerous neologicists have disagreed with Frege's assessment of his own project. Instead, they have tried to repair logicism by creating new formulations of arithmetic that do not succumb to Russell's paradox. In turn, most of these attempts failed for their own reasons or simply proved unattractive and untenable. But there is a third way: dialetheism. Perhaps simply accepting Russell's paradox – accepting that a statement can be both true and false – could save the logicist project. This path is initially attractive because it is a simple response to paradox. Rather than rejecting the Russell set because it both is and is not a member of itself, dialetheism admits inconsistent sets. Through this route, an unrestricted comprehension schema can be (re-)established as follows: "for any condition or property, including paradoxical ones like non-self-membership, there exists a corresponding set."[6] This is possible because the motivation for restricting comprehension at all is in order to avoid inconsistent sets. In other words, the reason

[5] For the sake of brevity, in this paper I use modern set-theoretic notation. For a more detailed account of how Russell's paradox emerges using Frege's language, see Shapiro, *Thinking about Mathematics*, 114–15.
[6] Priest and Berto, "Dialetheism."

a naive set theory is rejected – and much of the logicist project is abandoned – is because contradictions are rejected. Can dialetheism, which accepts contradictions, avoid this problem?

The first order of business is to clarify what dialetheism really is. It is important to emphasize that dialetheism does *not* mean believing all contradictions. Some contradictions must be false; if not, triviality and irrationalism result.[7] For example, a dialetheist will *not* endorse the statement "I am a fried egg and I am not a fried egg" because the sentence "I am a fried egg" is simply irrational. Similarly, dialethism is not an argument for accepting sentences like "$0 = 1$ & $0 \neq 1$" or "the Earth orbits and does not orbit the Sun." In short, a dialetheist rejects "bad" contradictions. What is a "bad" contradiction? In the words of Alan Weir, this is "one of the most fundamental difficulties with dialetheism."[8] It certainly seems that the distinction between "good" and "bad" contradictions is mostly based on intuition. What most dialetheists attempt is to retain a normal picture of the world except in the few cases where a contradiction seems palatable. In other words, dialetheists do *not* want to be irrational, but *are* willing to bite the bullet when confronted with a paradox that seems to result in contradiction.

Setting aside these issues for the moment, I focus now on what dialetheists *can* agree on. In the words of JC Beall, this is the denial that "the intersection of truth and falsity is necessarily empty."[9] By most dialetheists' views, sentences like Russell's paradox and the liar paradox belong in this category. What is the truth value of these contradictions, then? The answer could be that they are both true and false; neither true nor false, but some third value; or true to some degree (to illustrate just a few options). Further developing this question leads to paraconsistent logic. There are

[7] These are almost universally held to be undesirable, but exactly why triviality and irrationalism are so objectionable is harder to pin down.
[8] Weir, "There Are No True Contradictions," 388.
[9] Beall, "Introduction: At the Intersection," 3.

actually a number of these logical systems, many of which form novel and viable theoretical perspectives. Here, it is important to note that paraconsistency and dialetheism are *not* synonymous. As Beall notes, "any rational version of the latter will require the former, but the converse seems not to hold."[10] So what does paraconsistency confront that dialetheism does not?

The answer is "explosion." This principle of classical logic – in more rarefied circles, *ex contradictione quodlibet* ("from contradiction, anything [follows]") – is the most commonly invoked objection to contradiction and hence to dialetheism itself. It can be stated as follows:

(1) Suppose $A \wedge \neg A$ is true (a contradiction).

(2) Then $A \vee B$ is true, since A is true by (1).

(3) But $\neg A$ is also true by (1), so (2) is false unless B is true.

(4) Hence, B is true.

Here, A and B can stand for anything. In particular, from any true contradiction *any* statement B can be proven true – including, say, "there are five unicorns in my bedroom" or "the sky is green."[11] This is, needless to say, undesirable: it becomes trivial that everything is true. Explosion is the most common explanation for why contradictions are bad in "classical" logic.[12] The attempt to avoid explosion led to the development of paraconsistent logics.

How does such a logic avoid explosion? To explain, I use Graham Priest's "model-theoretic account of one propositional paraconsistent logic."[13] Priest considers classical logic to evaluate

[10] Ibid., 7.
[11] The only "restriction" on *B* is that it must be a sentence that is truth-evaluable – so *B* cannot be nonsensical – say, "crow electric green buffalo" – or non-assertoric – say, "what time is it?". I thank Joshua Schechter for clarifying this point.
[12] Interestingly, Priest demonstrates that the principle of explosion only became entrenched "in the second half of the nineteenth century." See Priest, "What's so Bad about Contradictions?," 25.
[13] Ibid.

formulas through functions that assign a value of 1 (true) or 0 (false). For example, "Rhode Island is an island <u>or</u> is not an island" returns a value of 1, while "Rhode Island is an island <u>and</u> is not an island" returns a value of 0. To move away from this logic, Priest defines a relation R that takes a formula α and relates it to 1, 0, both, or neither – in the form $R(\alpha, 1)$. For example, R("Rhode Island is an island <u>or</u> is not an island", 1) as above. If one were to use paraconsistent logic to encode dialetheism, one could give R("Rhode Island is an island <u>and</u> is not an island", 1).[14] Most parameters (including conjunction, disjunction, and negation) are similar to those in classical logic. Logical inference is also similar: "an inference is valid iff whenever the premises are true, so is the conclusion."[15] This statement requires some explication, since it is key to avoiding explosion. Take a formula α and a formula β and let $R(\alpha, 1)$ and $R(\beta, 0)$. Let α be the premises and β be the conclusion. By definition, whenever α is true β is false. Therefore, the inference is not valid. On the other hand, if $R(\beta, 1)$ and $R(\alpha, 1)$ the inference is valid. Now, let $\alpha = A \wedge \neg A$. Once again, the inference is not valid. This shows how dialetheism can be encoded in this paraconsistent logic. To give a clear example, consider the following case. Let R("Rhode Island is an island", 1) and R("Rhode Island is an island", 0) and R("the sky is green", 0). Whenever "Rhode Island is an island" (the premise) is true, "the sky is green" (the conclusion) is false – because it is always the case that R("the sky is green", 0). By Priest's principle, this inference is invalid. A contradiction has been stated while avoiding explosion.

In sum, this form of paraconsistent logic preserves the rules of inference (and other principles of classical logic) while avoiding explosion. Rather than two possible values for evaluations – true and false – we now have four: true, false, neither true nor false, and both true and false. The first

[14] Incidentally, this is a good example of an empirical contradiction that does seem to be true. These kinds of statements are another motivation for dialetheism. See ibid., 28.
[15] Ibid., 26.

two possibilities are very familiar. Indeed, the third option is also common. For example, Aristotle believed that contingent statements about the future are neither true nor false. Modern logic has provided other examples, too: undecidable sentences in mathematics and science, category errors, statements that are not truth-evaluable, and so on. The most intriguing part of this paraconsistent logic is what lies in the intersection of truth and falsity. As noted before, the most plausible candidates are contradictions that result from self-referential paradoxes. It should be clear that a paraconsistent logic is viable and consistent with dialetheism.

At last, I return to the question I posed earlier: does dialetheism resolve Russell's paradox and hence help the logicist project? To answer, I apply the paraconsistent logic outlined above. Let r be the statement of Russell's paradox. Then $R(r, 1)$ and $R(r, 0)$. By the rules of inference outlined above, for any β such that $R(\beta, 0)$ the inference is invalid. In short, explosion does not hold. Therefore, accepting the contradiction that results from Russell's paradox does not result in inconsistency and catastrophe. Most importantly, a paraconsistent logic allows us to accept contradictions while preserving almost all of classical logic. One important result is that naive set theory – including naive comprehension – can be fully reconstructed.[16] This goes a long way towards recovering the logicist project. In the framework of this paraconsistent logic, "there is no obvious reason why one should not take the view that naive set theory provides an adequate foundation for mathematics, and that naive set theory is reducible to logic via the naive comprehension schema."[17] In short, logicism is not necessarily doomed.

I now briefly sketch some reasons to favor and to disfavor a paraconsistent logicism. Graham Priest, of course, argues for this project. His main reasoning is that "inconsistent objects like the

[16] Priest, *In Contradiction*, 247–60.
[17] Mortensen, "Inconsistent Mathematics."

Russell set" are merely the latest in a long line of shocking mathematical discoveries that have included the irrationals, imaginary numbers, and transfinite sets. In each of these previous cases, mathematics has benefited by incorporating these new objects while preserving "the central parts of previous mathematical thought."[18] This argument is strengthened by the example of Zach Weber and Maarten McKubre-Jordens, who showed that "by weakening the logic within which we work, so as to allow for the possibility of non-trivial inconsistency, we are still able to do everyday mathematics."[19] In a later paper, Weber demonstrated an even more striking result: paraconsistent logic (with dialetheism) "provides a great deal more access into the transfinite than any classical counterpart." In particular, in the naive set theory that this system allows, many vexing problems – including the well-ordering problem and the continuum hypothesis – "are all settled."[20] In sum, not only does dialetheism allow for the reconstruction of logicism and naive set theory, but such a program has real, valuable mathematical results.

So far, it seems dialetheism has received a ringing endorsement. Yet there remain deep semantic and metaphysical questions. First, let me return to the issue of "bad" contradictions, as mentioned above. How do we distinguish "bad" contradictions from "good" dialetheias? Certainly, dialetheism doesn't argue *for* the idea that everything is both true and false. Yet does it provide any defense against such a claim? Priest presents many arguments against this form of trivialism, but Frederick Kroon argues that "the problem may well be inescapable if the dialetheist is also a realist."[21] The resolution given by Kroon is fictionalism: respect the talk of contradictions, but reject that it says anything about the world. Another issue for dialetheism is what to make of reason. Current reasoning often proceeds as follows: I hold "I am a pickled egg"; I have sufficient

[18] Priest, *In Contradiction*, 260.
[19] McKubre-Jordens and Weber, "Real Analysis in Paraconsistent Logic," 901.
[20] Weber, "Transfinite Cardinals in Paraconsistent Set Theory," 269.
[21] Kroon, "Realism and Dialetheism," 247.

evidence to believe ¬("I am a pickled egg"); hence, I believe I am not a pickled egg. In a dialetheic world, what argument is there against merely holding both things to be true? In other words, why would dialetheism endorse this kind of "normal" reasoning? If dialetheism cannot provide sufficient grounds for such reasoning, it fails a very important function of logic. Finally, I would like to note that in this paper I have completely ignored Gödel's two incompleteness theorems. An argument for dialetheism must confront and account for these results.

In conclusion, dialetheism – especially in the form of Graham Priest's paraconsistent logic – is a valuable and viable theoretical perspective. It is a natural result of the discovery of self-referential paradoxes and alleviates worries caused by the contradictions arising from these paradoxes. Although a classical logician might worry about explosion, paraconsistent logic allows the development of dialetheism while avoiding this result. Furthermore, this system is useful for interests beyond the purely mathematical. Although there are still significant issues to worry about, dialetheism is clearly both valuable and viable.

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